Lorentz transformations without the second postulate of relativity.

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Contents

1 The first and second law of Newton.

The first and second law of Newton are perhaps the most important laws in physics. These laws are given by

1. The first law of Newton states that a body on which no force is acting has uniform momentum.
2. The second law of Newton states that the force acting on a body is the change in the momentum of the body with respect to time.

The second law of Newton can be written as $\vec{F} = \dfrac{d\vec{p}}{dt}$, where $\vec{F}$ is the force acting of the body and $\vec{p}$ is the momentum of the body. The momentum $\vec{p}$ is defined as $\vec{p} = m\vec{v}$, where m is the mass of the body and $\vec{v}$ is the velocity of the body. We may however wonder *what is a force acting on a body?* And we may say that *a force acting on a body is the change of the momentum of the body with respect to time.* In that sense, the second law of Newton is not actually a law, but rather a definition; it is the definition of the force. We may also wonder *are the laws of Newton valid in all systems of reference?* We may say that *the laws of Newton are valid in those systems of reference for which a body on which no force is acting has a uniform momentum.* In that sense, the first law of Newton is not actually a law, but rather the definition of the system of reference in which the laws of Newton are valid.

We shall use the first law of Newton for two definitions of the systems of reference in which the laws of Newton are valid:

The weak inertial system of reference is a system of reference in which free moving bodies have uniform momentum.

The strong inertial system of reference is a system of reference in which free moving bodies have uniform momentum and uniform velocity.

If no force is acting on a body, then it is clear that $\frac{d\vec{p}}{dt} = 0$, where $\vec{p} = m\vec{v}$. We can always write $\vec{v} = v\hat{e}_v$, where $\hat{e}_v$ is a unit-vector, thus $\hat{e}_v \cdot \hat{e}_v = 1$. Then it is clear that $\hat{e}_v \cdot \frac{d\hat{e}_v}{dt} = 0$. Therefore we obtain $\left(\frac{dm}{dt}v + m\frac{dv}{dt}\right)\hat{e}_v + mv\frac{d\hat{e}_v}{dt} = 0$. Let us assume that the mass m nor the speed v are zero. Then it is clear that $\frac{d\hat{e}_v}{dt} = 0$ and $\frac{dm}{dt}v + m\frac{dv}{dt} = 0$. This means that the direction of the momentum and velocity does not change in time and that mv is constant. We shall give another definition:

A dynamical body is a body that has a mass which is either constant or depends on the speed of the body.

For a free moving dynamical body we obtain $\left(\frac{dm}{dv}v + m\right)\frac{dv}{dt} = 0$, therefore a free moving dynamical body has uniform speed. It is clear that whenever bodies are dynamical - then a weak inertial system of reference is also a strong inertial system of reference.

It is also clear that a body is not moving also satisfies the condition $\frac{d\vec{p}}{dt}$ as in this case $\vec{p} = 0$. Such a body is said to be at rest. Therefore we can extend our definitions for free bodies - bodies that are either at rest or moving. And whenever bodies are dynamical - there is no distinction between a weak inertial system of reference or a strong inertial system of reference. In this case we shall refer to an inertial system of reference. A small summary:

The mass hypothesis. All masses (and bodies) are dynamical, i.e. the mass of bodies is either constant or depends on the speed of the body.

The inertial system of reference definition. An inertial system of reference is any system of reference in which free bodies have uniform momentum. As bodies are dynamical, free bodies also have uniform velocity.

We shall continue the analysis using the mass hypothesis and the inertial system of reference definition.

2 Transformations between inertial systems of reference.

Let us consider two systems of reference $\mathfrak{S}$ and $\mathfrak{S}'$. Each system is used to describe bodies using *space* and *time*. The co-ordinates (t, x_k) are used for $\mathfrak{S}$ and the co-ordinates $(t\, x'_\ell)$ are used for $\mathfrak{S}'$. The general co-ordinate transformations from $\mathfrak{S}$ to $\mathfrak{S}'$ can be written as

$$\left[\begin{array}{lcl} t' & = & t'(t, x_k) \\ \\ x'_\jmath & = & x'_\jmath(t, x_k) \end{array}\right. \tag{1}$$

It is clear that

$$\frac{dx'_\jmath}{dt'} \quad = \quad \left(\frac{\partial x'_\jmath}{\partial x_k}\frac{dx_k}{dt} + \frac{\partial x'_\jmath}{\partial t}\right) \Big/ \left(\frac{\partial t'}{\partial x_k}\frac{dx_k}{dt} + \frac{\partial t'}{\partial t}\right). \tag{2}$$

A body $\mathcal{B}_o$ at rest at the origin of $\mathfrak{S}$ is a free body. For this body we can write $\frac{dx_k}{dt} = 0$, therefore

$$\frac{dx'_\jmath}{dt'} \quad = \quad \left(\frac{\partial x'_\jmath}{\partial t}\right) \Big/ \left(\frac{\partial t'}{\partial t}\right) \tag{3}$$

for the body $\mathcal{B}_o$ as described in $\mathfrak{S}'$. As the body is free in $\mathfrak{S}$ it is also free in $\mathfrak{S}'$. Therefore the components $\frac{dx'_\jmath}{dt'}$ are constant, thus

$$\frac{\partial^2 x'_\jmath}{\partial t^2}\frac{\partial t'}{\partial t} - \frac{\partial x'_\jmath}{\partial t}\frac{\partial^2 t'}{\partial t^2} = 0. \tag{4}$$

Then we obtain $\frac{\partial^2 x'_\jmath}{\partial t^2} = 0$ and $\frac{\partial^2 t'}{\partial t^2} = 0$, so

$$\begin{aligned} \frac{\partial^2 x'_\jmath}{\partial t^2} = 0 \quad &\Rightarrow \quad \frac{\partial x'_\jmath}{\partial t} = A_\jmath \\ &\Rightarrow \quad x'_\jmath = A_\jmath t + B_\jmath. \end{aligned} \tag{5a}$$

and

$$\begin{aligned} \frac{\partial^2 t'}{\partial t^2} = 0 \quad &\Rightarrow \quad \frac{\partial t'}{\partial t} = A \\ &\Rightarrow \quad t' = At + B. \end{aligned} \tag{5b}$$

So the transformation of $x'_\jmath$ and t' are linear in t. We can also consider a body $\mathcal{B}'_o$ at rest at the origin of $\mathfrak{S}'$. The body $\mathcal{B}'_o$ is a free body. For this body we can write $\frac{dx_\jmath}{dt} = 0$, therefore

$$\frac{\partial x'_\jmath}{\partial x_k}\frac{dx_k}{dt} + \frac{\partial x'_\jmath}{\partial t} \quad = \quad 0. \tag{6}$$

As the body is free in $\mathfrak{S}$ it is also free in $\mathfrak{S}$. Therefore the components $\dfrac{dx_k}{dt}$ are constant, thus

$$\frac{\partial^2 x'_j}{\partial x_\ell \partial x_k}\frac{dx_k}{dt}\frac{dx_\ell}{dt}+\frac{\partial^2 x'_j}{\partial t \partial x_k}\frac{dx_k}{dt}+\frac{\partial^2 x'_j}{\partial x_\ell \partial t}\frac{dx_\ell}{dt}+\frac{\partial^2 x'_j}{\partial t^2} \quad = \quad 0, \tag{7}$$

valid for arbitrary values of the components $\dfrac{dx_k}{dt}$. We obtain $\dfrac{\partial^2 x'_j}{\partial x_\ell \partial x_k} = 0$, $\dfrac{\partial^2 x'_j}{\partial t \partial x_k} = 0$ and $\dfrac{\partial^2 t'}{\partial t^2} = 0$, so

$$\begin{aligned} \frac{\partial^2 x'_j}{\partial x_\ell \partial x_k} = 0 \quad &\Rightarrow \quad \frac{\partial x'_j}{\partial x_k} = A_j^k \\ &\Rightarrow \quad x'_j = A_j^k x_k + A_j. \end{aligned} \tag{8}$$

So the transformation of x'_j is linear in x_k. The other relations do not add information to the transformations. We obtain

$$\left[\begin{array}{lcl} t' &=& At + B \\ \\ x'_j &=& A_j^k x_k + A_j t \end{array}\right. \tag{9}$$

For the inverse for x_k we obtain

$$x_k \quad = \quad \left(A^{-1}\right)_k^j x'_j - \frac{\left(A^{-1}\right)_k^j A_j}{A} t + \frac{\left(A^{-1}\right)_k^j A_j}{A} B. \tag{10}$$

But this should take the form $x_k = A'^j_k x'_j + A'_k t'$ therefore B is linear in space. Whence

$$\left[\begin{array}{lcl} t' &=& A^k x_k + At \\ \\ x'_j &=& A_j^k x_k + A_j t \end{array}\right. \tag{11}$$

The transformations between inertial systems of reference are linear transformations. It is also clear that the origin of $\mathfrak{S}'$ is described by $x'_j = 0$. The origin of $\mathfrak{S}'$ is described in $\mathfrak{S}$ by the equations $A_j^k x_k + A_j t = 0$, so $x_k = -\left(A^{-1}\right)_k^j A_j t$. This gives a uniform velocity $x_k = v_k t$, where $v_k = -\left(A^{-1}\right)_k^j A_j$. Whence inertial systems of reference have a relative uniform velocity.

3 The transformation group.

Let us consider several inertial systems of reference $\mathfrak{S}_\kappa$. Each inertial system of reference is a collection of events, where an event is no more than a space-time co-ordinate. We shall denote an event as ξ. The only knowledge that we have about two inertial systems of reference is the relative velocity between the inertial systems of reference. Spacial orientations can be ignored. The relative velocity between $\mathfrak{S}_\kappa$ and $\mathfrak{S}_\lambda$ is denoted as $\vec{v}_{\kappa\lambda}$, it describes the velocity of the origin of $\mathfrak{S}_\kappa$ as described within $\mathfrak{S}_\lambda$. The co-ordinate transformation from $\mathfrak{S}_\kappa$ to $\mathfrak{S}_\lambda$ can be written as $\mathbf{T}_{\kappa\lambda}$. We may write $\mathbf{T}_{\kappa\lambda} : \mathfrak{S}_\kappa \ni \xi \mapsto \mathbf{T}_{\kappa\lambda}\xi \in \mathfrak{S}_\lambda$. However, this transformation depends only on the relative velocity between $\mathfrak{S}_\kappa$ and $\mathfrak{S}_\lambda$ - we may write $\mathbf{T}_{\kappa\lambda} = \mathbf{T}(\vec{v}_{\kappa\lambda})$, where $\vec{v}_{\kappa\lambda}$ is defined as the velocity of the origin of $\mathfrak{S}_\lambda$ with respect to $\mathfrak{S}_\kappa$. The collection $\mathfrak{T}$ is a group as:

1. **Closed.** The mapping $\mathfrak{S}_\kappa \mapsto \mathfrak{S}_\lambda \mapsto \mathfrak{S}_\mu$ can also be written as $\mathfrak{S}_\kappa \mapsto \mathfrak{S}_\mu$. Then it is clear that $\mathbf{T}_{\kappa\lambda}\mathbf{T}_{\lambda\mu} = \mathbf{T}_{\kappa\mu}$ for all $\mathbf{T}_{\kappa\lambda}, \mathbf{T}_{\lambda\mu}, \mathbf{T}_{\kappa\mu} \in \mathfrak{T}$. The collection $\mathfrak{T}$ is closed. We may also write $\mathbf{T}(\vec{v}_\jmath)\mathbf{T}(\vec{v}_k) = \mathbf{T}(\vec{v}_\jmath \oplus \vec{v}_k)$.

2. **Unity.** The mapping $\mathfrak{S}_\kappa \mapsto \mathfrak{S}_\kappa \mapsto \mathfrak{S}_\lambda$ can also be written as $\mathfrak{S}_\kappa \mapsto \mathfrak{S}_\lambda$. Then it is clear that $\mathbf{T}_{\kappa\kappa}\mathbf{T}_{\kappa\lambda} = \mathbf{T}_{\kappa\lambda}$ for all $\mathbf{T}_{\kappa\kappa}, \mathbf{T}_{\kappa\lambda} \in \mathfrak{T}$, i.e. $\mathbf{T}_{\kappa\kappa} = \mathbf{1}$. The collection $\mathfrak{T}$ has a unit element. As $\mathbf{T}_{\kappa\kappa} = \mathbf{T}(\vec{v}_{\kappa\kappa})$ and $\vec{v}_{\kappa\kappa} = \vec{0}$, we obtain that $\mathbf{T}(\vec{0}) = \mathbf{1}$.

3. **Inverse.** The mapping $\mathfrak{S}_\kappa \mapsto \mathfrak{S}_\lambda \mapsto \mathfrak{S}_\kappa$ can also be written as $\mathfrak{S}_\kappa \mapsto \mathfrak{S}_\kappa$. Then it is clear that $\mathbf{T}_{\kappa\lambda}\mathbf{T}_{\lambda\kappa} = \mathbf{1}$ for all $\mathbf{T}_{\kappa\lambda}, \mathbf{T}_{\lambda\kappa} \in \mathfrak{T}$. As $\mathbf{T}_{\lambda\kappa} \in \mathfrak{T}$ once $\mathbf{T}_{\kappa\lambda} \in \mathfrak{T}$, each transformation has an inverse. So $\mathbf{T}(\vec{v}_{\kappa\lambda})\mathbf{T}(\vec{v}_{\lambda\kappa}) = \mathbf{1}$, so - as $\mathbf{T}(\vec{v}_{\kappa\lambda})\mathbf{T}(\vec{v}_{\lambda\kappa}) = \mathbf{T}(\vec{v}_{\kappa\lambda} \oplus \vec{v}_{\lambda\kappa})$ - we obtain that $\vec{v}_{\kappa\lambda} \oplus \vec{v}_{\lambda\kappa} = \vec{0}$.

We now consider the subset $\mathfrak{T}_x$ of $\mathfrak{T}$ that is formed by transformations between inertial systems of reference that all are in relative motion along the $x-$axis. We may now write $\mathbf{T}(v_{\kappa\lambda})$. It is clear that $\mathbf{T}(v_\kappa)\mathbf{T}(v_\lambda) = \mathbf{T}(v_\kappa \oplus v_\lambda)$, however, in general we have $v_\kappa \oplus v_\lambda \neq v_\kappa + v_\lambda$. As the collection $\mathfrak{T}$ is a group, we may consider group-parameters. As there is one parameter we only have one group-parameter. Let this group-parameter be denoted as ζ. Then we have $\mathbf{T}_o(\zeta_\kappa)\mathbf{T}_o(\zeta_\lambda) = \mathbf{T}_o(\zeta_\kappa + \zeta_\lambda)$ and $\mathbf{T}_o(\zeta_\kappa) = \mathbf{T}(v_\kappa)$. This group-parameter ζ is related with the velocity v, written as $\zeta = \phi(v)$. so we obtain $\phi(v_1 \oplus v_2) = \phi(v_1) + \phi(v_2)$. Note that ϕ is a bijective function, thus ϕ^{-1} is defined. This also gives the general form of velocity compositions as

$$v_1 \oplus v_2 = \phi^{-1}\left(\phi(v_1) + \phi(v_2)\right). \tag{12}$$

As $\mathbf{T}_o(\zeta_\kappa)\mathbf{T}_o(\zeta_\lambda) = \mathbf{T}_o(\zeta_\kappa + \zeta_\lambda)$, it is clear that $\mathbf{T}_o(\zeta) = \mathbf{T}_o{}^\zeta$. The general linear transformation can be written as

$$\mathbf{T}^{[4\times4]} = \begin{pmatrix} f & \ell_1 & \ell_2 & \ell_3 \\ k_1 & q_{11} & q_{12} & q_{13} \\ k_2 & q_{21} & q_{22} & q_{23} \\ k_3 & q_{31} & q_{32} & q_{33} \end{pmatrix}. \tag{13}$$

However, we can always consider spatial orientation such that $q_{k\ell} = 0$ if $k \neq \ell$ and that $k_2 = 0$ and $k_3 = 0$. As this form is valid for the inverse as well, we obtain the simpler form

$$\mathbf{T}^{[4\times4]} = \begin{pmatrix} f & \ell_1 & 0 & 0 \\ k_1 & q_{11} & 0 & 0 \\ 0 & 0 & q_{22} & 0 \\ 0 & 0 & 0 & q_{33} \end{pmatrix}. \tag{14}$$

So we may write $\mathbf{T}^{[4\times4]} = \mathbf{T}_{tx}^{[2\times2]} \oplus \mathbf{T}_{y}^{[1\times1]} \oplus \mathbf{T}_{z}^{[1\times1]}$. Therefore we can write $\left(\mathbf{T}^{[4\times4]}\right)^{\zeta} = \left(\mathbf{T}_{tx}^{[2\times2]}\right)^{\zeta} \oplus \left(\mathbf{T}_{y}^{[1\times1]}\right)^{\zeta} \oplus \left(\mathbf{T}_{z}^{[1\times1]}\right)^{\zeta}$. We now need to find $\left(\mathbf{T}_{tx}^{[2\times2]}\right)^{\zeta}$.

Let us consider

$$\mathbf{T}_o = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}. \tag{15}$$

First we note that

$$\begin{aligned}
{\mathbf{T}_o}^2 &= \begin{pmatrix} \tau_{11}^2 + \tau_{12}\tau_{21} & \tau_{11}\tau_{12} + \tau_{12}\tau_{22} \\ \tau_{21}\tau_{11} + \tau_{22}\tau_{21} & \tau_{21}\tau_{12} + \tau_{22}^2 \end{pmatrix} \\
&= (\tau_{11} + \tau_{22}) \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} - (\tau_{11}\tau_{22} - \tau_{12}\tau_{21}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= (\tau_{11} + \tau_{22})\, \mathbf{T}_o - (\tau_{11}\tau_{22} - \tau_{12}\tau_{21})\, \mathbf{1}.
\end{aligned}$$

The *trace* of the matrix $\mathbf{T}_o$ is defined as $\mathrm{tr}(\mathbf{T}_o) = \tau_{11} + \tau_{22}$. The *determinant* of the matrix $\mathbf{T}_o$ is defined as $\det(\mathbf{T}_o) = \tau_{11}\tau_{22} - \tau_{12}\tau_{21}$. The trace is denoted as χ and the determinant is denoted as Δ. So we may write

$${\mathbf{T}_o}^2 = \chi \mathbf{T}_o - \Delta \mathbf{1}. \tag{16}$$

The matrix $\mathbf{T}_o$ has eigenvalues λ defined by $\det(\mathbf{T}_o - \lambda \mathbf{I}) = 0$. This gives the equation $(\tau_{11} - \lambda)(\tau_{22} - \lambda) - \tau_{12}\tau_{21} = 0$, then

$$\lambda_\pm = \chi/2 \pm \sqrt{\chi^2/4 - \Delta}. \tag{17}$$

Note that

$$\lambda_\pm \lambda_\mp = \Delta \tag{18a}$$

and

$$\lambda_\pm + \lambda_\mp = \chi. \tag{18b}$$

So we can write

$${\mathbf{T}_o}^2 = (\lambda_\pm + \lambda_\mp)\, \mathbf{T}_o - \lambda_\pm \lambda_\mp \mathbf{1}. \tag{19}$$

We define

$$\mathbf{T}_{o\pm} = \mp (\mathbf{T}_o - \lambda_\pm \mathbf{1}). \tag{20}$$

Then we can write

$$\mathbf{1} = \frac{1}{\lambda_+ - \lambda_-} \mathbf{T}_{o-} + \frac{1}{\lambda_+ - \lambda_-} \mathbf{T}_{o+}. \tag{21a}$$

and

$$\mathbf{T}_o = \frac{\lambda_+}{\lambda_+ - \lambda_-} \mathbf{T}_{o-} + \frac{\lambda_-}{\lambda_+ - \lambda_-} \mathbf{T}_{o+}. \tag{21b}$$

It is clear that

$$\begin{aligned}
\mathbf{T}_{o\pm}\mathbf{T}_{o\mp} &= -\left(\mathbf{T}_o - \lambda_\pm \mathbf{1}\right)\left(\mathbf{T}_o - \lambda_\mp \mathbf{1}\right) \\
&= {\mathbf{T}_o}^2 - \left(\lambda_\pm + \lambda_\mp\right)\mathbf{T}_o + \lambda_\pm\lambda_\mp \mathbf{1} \\
&= \left(\lambda_\pm + \lambda_\mp\right)\mathbf{T}_o - \lambda_\pm\lambda_\mp \mathbf{1} - \left(\lambda_\pm + \lambda_\mp\right)\mathbf{T}_o + \lambda_\pm\lambda_\mp \mathbf{1},
\end{aligned}$$

so

$$\mathbf{T}_{o\pm}\mathbf{T}_{o\mp} = 0. \tag{22a}$$

It is also clear that

$$\begin{aligned}
{\mathbf{T}_{o\pm}}^2 &= \left(\mathbf{T}_o - \lambda_\pm \mathbf{1}\right)^2 \\
&= {\mathbf{T}_o}^2 - 2\lambda_\pm \mathbf{T}_o + {\lambda_\pm}^2 \mathbf{1} \\
&= \left(\lambda_\pm + \lambda_\mp\right)\mathbf{T}_o - \lambda_\pm\lambda_\mp \mathbf{1} - 2\lambda_\pm \mathbf{T}_o + {\lambda_\pm}^2 \mathbf{1} \\
&= \left(\lambda_+ - \lambda_-\right) \cdot \mp \left(\mathbf{T}_o - \lambda_\pm \mathbf{1}\right),
\end{aligned}$$

so

$${\mathbf{T}_{o\pm}}^2 = \left(\lambda_+ - \lambda_-\right)\mathbf{T}_{o\pm}. \tag{22b}$$

Therefore

$${\mathbf{T}_{o\pm}}^n = \left(\lambda_+ - \lambda_-\right)^{n-1}\mathbf{T}_{o\pm}, \tag{23}$$

where n is a positive integer. As $\mathbf{T}_{o\pm}\mathbf{T}_{o\mp} = 0$ we can write

$${\mathbf{T}_o}^n = \left(\frac{\lambda_+}{\lambda_+ - \lambda_-}\right)^n {\mathbf{T}_{o-}}^n + \left(\frac{\lambda_-}{\lambda_+ - \lambda_-}\right)^n {\mathbf{T}_{o+}}^n,$$

so

$${\mathbf{T}_o}^n = \frac{{\lambda_+}^n}{\lambda_+ - \lambda_-}\mathbf{T}_{o-} + \frac{{\lambda_-}^n}{\lambda_+ - \lambda_-}\mathbf{T}_{o+}. \tag{24}$$

Using the definition $\mathbf{T}_{o\pm} = \mp\left(\mathbf{T}_o - \lambda_\pm \mathbf{1}\right)$ we obtain

$${\mathbf{T}_o}^n = \frac{{\lambda_+}^n - {\lambda_-}^n}{\lambda_+ - \lambda_-}\mathbf{T}_o - \lambda_+\lambda_- \frac{{\lambda_+}^{n-1} - {\lambda_-}^{n-1}}{\lambda_+ - \lambda_-}\mathbf{1}, \tag{25}$$

where n is a positive integer. We can easy see that this is valid for any integer and we can extend this to

$${\mathbf{T}_o}^\zeta = \frac{{\lambda_+}^\zeta - {\lambda_-}^\zeta}{\lambda_+ - \lambda_-}\mathbf{T}_o - \lambda_+\lambda_- \frac{{\lambda_+}^{\zeta-1} - {\lambda_-}^{\zeta-1}}{\lambda_+ - \lambda_-}\mathbf{1}. \tag{26}$$

We have $\lambda_+\lambda_- = \Delta$, so we can always write $\lambda_\pm = \sqrt{\Delta}\exp(\pm\psi)$. Therefore ${\lambda_+}^\xi - {\lambda_-}^\xi = 2\sqrt{\Delta}^\xi \sinh(\psi\xi)$ and we can write

$${\mathbf{T}_o}^\zeta = \sqrt{\Delta}^\zeta \left\{ \frac{1}{\sqrt{\Delta}} \frac{\sinh(\psi\zeta)}{\sinh(\psi)}\mathbf{T}_o - \frac{\sinh(\psi\zeta - \psi)}{\sinh(\psi)}\mathbf{1} \right\}. \tag{27}$$

It is also clear that $\lambda_+ + \lambda_- = 2\sqrt{\Delta}\cosh(\psi)$, thus $\chi = 2\sqrt{\Delta}\cosh(\psi)$. As $\chi = \tau_{11} + \tau_{22}$, we can always write

$$\tau_{11} = \sqrt{\Delta}\left(\cosh(\psi) + \mu\sinh(\psi)\right) \quad \text{and} \quad \tau_{22} = \sqrt{\Delta}\left(\cosh(\psi) - \mu\sinh(\psi)\right).$$

As $\Delta = \tau_{11}\tau_{22} - \tau_{12}\tau_{21}$, we find $\tau_{12}\tau_{21} = \Delta\sinh^2(\psi)\left(1 - \mu^2\right)$, so we can always write

$$\tau_{12} = \sqrt{\Delta}\sigma^{-1}\sinh(\psi)\left[1 - \mu\right] \quad \text{and} \quad \tau_{21} = \sqrt{\Delta}\sigma\sinh(\psi)\left[1 + \mu\right].$$

Using these expressions for $\tau_{..}$ we obtain

$$\mathbf{T}_o = \sqrt{\Delta}\cosh(\psi)\begin{pmatrix} 1 + \mu\tanh(\psi) & \sigma^{-1}\tanh(\psi)\left[1 - \mu\right] \\ \sigma\tanh(\psi)\left[1 + \mu\right] & 1 - \mu\tanh(\psi) \end{pmatrix}. \quad (28)$$

The final expression for ${\mathbf{T}_o}^{\zeta}$ is given by

$$\mathbf{T}_o(\zeta) = \sqrt{\Delta}^{\zeta}\cosh(\zeta)\begin{pmatrix} 1 + \mu\tanh(\zeta) & \frac{1}{\sigma}\tanh(\zeta)\left[1 - \mu\right] \\ \sigma\tanh(\zeta)\left[1 + \mu\right] & 1 - \mu\tanh(\zeta) \end{pmatrix}, \quad (29)$$

where ζ is the group-parameter; Δ, σ and μ are constants. This is the general transformation between inertial systems of reference that follows from the definition of the inertial system of reference itself, it does not require the second postulate of relativity. Classical physics is embedded as we obtain the Galileo transformations once we set $\Delta = 1$, $\sigma = \infty$ and $\mu = 0$. Relativity is embedded as we obtain the Lorentz transformations once we set $\Delta = 1$, $\sigma = c$ and $\mu = 0$. Both classical physics and relativity have in common that $\Delta = 1$ and $\mu = 0$. What kind of physics do we obtain once $\Delta \neq 1$ and/or $\mu \neq 0$?

We search for invariant speeds. Let v_o be an invariant speed, then we find $x' = v_o t'$ for $x = v_o t$. The co-ordinate transformation can be written as

$$\left[\begin{array}{lcl} \sigma t' &=& \sqrt{\Delta}^{\zeta}\cosh(\zeta)\left[[1 + \mu\tanh(\zeta)]\sigma t + \tanh(\zeta)[1 - \mu]x\right] \\ x' &=& \sqrt{\Delta}^{\zeta}\cosh(\zeta)\left[\tanh(\zeta)[1 + \mu]\sigma t + [1 - \mu\tanh(\zeta)]x\right] \end{array}\right. . \quad (30)$$

When we put in $x = \beta\sigma t$, then find

$$\left[\begin{array}{lcl} \sigma t' &=& \sqrt{\Delta}^{\zeta}\cosh(\zeta)\left[1 + \mu\tanh(\zeta) + \tanh(\zeta)[1 - \mu]\beta\right]\sigma t \\ x' &=& \sqrt{\Delta}^{\zeta}\cosh(\zeta)\left[\tanh(\zeta)[1 + \mu] + [1 - \mu\tanh(\zeta)]\beta\right]\sigma t \end{array}\right. . \quad (31)$$

When the speed $\beta\sigma$ is invariant then we can write $x' = \beta\sigma t'$, therefore

$$\beta \quad = \quad \frac{\tanh(\zeta)[1+\mu] + [1-\mu\tanh(\zeta)]\beta}{1+\mu\tanh(\zeta) + \tanh(\zeta)[1-\mu]\beta}. \tag{32}$$

Then we obtain

$$\beta \quad = \quad \pm\frac{1\mp\mu}{1-\mu}. \tag{33}$$

This means that there are two invariant speeds, a forward invariant speed and a backward invariant speed. Forward and backward are defined with respect to the $x-$ axis. When we reverse the orientation along the $x-$ axis we obtain

$$\beta \quad = \quad \pm\frac{1\pm\mu}{1-\mu}. \tag{34}$$

This is only consistent if $\mu = 0$. It is then clear that σ is the universal velocity, it is the same in all inertial systems of reference. The general form of the co-ordinate transformation can be written as

$$\mathbf{T}_{tx}^{[2\times2]} \quad = \quad \sqrt{\Delta}^{\zeta}\begin{pmatrix} \cosh(\zeta) & \frac{1}{\sigma}\sinh(\zeta) \\ \sigma\sinh(\zeta) & \cosh(\zeta) \end{pmatrix}. \tag{35}$$

And eventually we find

$$\mathbf{T}^{[4\times4]} \quad = \quad \exp(\phi\zeta)\begin{pmatrix} \cosh(\zeta) & \frac{1}{\sigma}\sinh(\zeta) & 0 & 0 \\ \sigma\sinh(\zeta) & \cosh(\zeta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{36}$$

The co-ordinate transformation can be written as

$$\left[\begin{array}{rcl} \sigma t' &=& \exp(\phi\zeta)\Big[\cosh(\zeta)\sigma t + \sinh(\zeta)x\Big] \\ x' &=& \exp(\phi\zeta)\Big[\sinh(\zeta)\sigma t + \cosh(\zeta)x\Big] \\ y' &=& \exp(\phi\zeta)y \\ z' &=& \exp(\phi\zeta)z \end{array}\right. \tag{37}$$

Then it is clear that

$$\left[\begin{array}{rcl} [\sigma t']^2 &=& \exp(2\phi\zeta)\Big[\cosh^2(\zeta)[\sigma t]^2 + 2\sinh(\zeta)\cosh(\zeta)x\sigma t + \sinh^2(\zeta)x^2\Big] \\ x'^2 &=& \exp(2\phi\zeta)\Big[\sinh^2(\zeta)[\sigma t]^2 + 2\sinh(\zeta)\cosh(\zeta)x\sigma t + \cosh^2(\zeta)x^2\Big] \\ y'^2 &=& \exp(2\phi\zeta)y^2 \\ z'^2 &=& \exp(2\phi\zeta)z^2 \end{array}\right.$$

Therefore

$$[\sigma t']^2 - x'^2 - y'^2 - z'^2 \quad = \quad \exp(2\phi\zeta)\Big([\sigma t]^2 - x^2 - y^2 - z^2\Big). \tag{38}$$

Let us define $s^2 = \sigma^2 t^2 - x^2 - y^2 - z^2$, then we have ${s'}^2 = \exp(2\phi\zeta)s^2$. As σ is the universal velocity it is clear that gives $s = 0$, and trivial that $s' = 0$. In case $\phi = 0$ then s is invariant.

It is also clear that $x' = 0$ for $x = vt$, where v is the speed of the origin of $\mathfrak{S}'$ with respect to $\mathfrak{S}$. Then we obtain

$$\tanh(\zeta) = -\frac{v}{\sigma}. \tag{39}$$

As $\cosh\zeta = \dfrac{1}{\sqrt{1-\tanh^2\zeta}}$ and $\sinh\zeta = \dfrac{\tanh(\zeta)}{\sqrt{1-\tanh^2\zeta}}$, we obtain

$$\cosh(\zeta) = \frac{1}{\sqrt{1-v^2/\sigma^2}} \quad \text{and} \quad \sinh(\zeta) = \frac{-v/\sigma}{\sqrt{1-v^2/\sigma^2}}. \tag{40}$$

The co-ordinate transformation can be written as

$$\left[\begin{array}{lcl} t' &=& \exp(\phi\zeta/\sigma)\left[\dfrac{1}{\sqrt{1-v^2/\sigma^2}}t - \dfrac{v/\sigma^2}{\sqrt{1-v^2/\sigma^2}}x\right] \\ x' &=& \exp(\phi\zeta/\sigma)\left[-\dfrac{v}{\sqrt{1-v^2/\sigma^2}}t + \dfrac{1}{\sqrt{1-v^2/\sigma^2}}x\right] \\ y' &=& \exp(\phi\zeta/\sigma)y \\ z' &=& \exp(\phi\zeta/\sigma)z \end{array}\right. \tag{41}$$

The case $\phi \to 0$ gives

$$\left[\begin{array}{lcl} t' &=& \dfrac{1}{\sqrt{1-v^2/\sigma^2}}t - \dfrac{v/\sigma^2}{\sqrt{1-v^2/\sigma^2}}x \\ x' &=& -\dfrac{v}{\sqrt{1-v^2/\sigma^2}}t + \dfrac{1}{\sqrt{1-v^2/\sigma^2}}x \\ y' &=& y \\ z' &=& z \end{array}\right. \tag{42}$$

known as the Lorentz transformations. The case $\sigma \to \infty$ gives

$$\left[\begin{array}{lcl} t' &=& t \\ x' &=& x - vt \\ y' &=& y \\ z' &=& z \end{array}\right. \tag{43}$$

known as the Galileo transformations. As ζ is the group-parameter it is clear that

$$\zeta(v_1 \oplus v_2) = \zeta(v_1) + \zeta(v_2). \tag{44}$$

And $\zeta(v) = \tanh^{-1}(-v/\sigma)$, so $\zeta(v) = \ln\left(\sqrt{\dfrac{\sigma+v}{\sigma-v}}\right)$, so we obtain

$$\frac{\sigma + [v_1 \oplus v_2]}{\sigma - [v_1 \oplus v_2]} = \frac{\sigma+v_1}{\sigma-v_1}\frac{\sigma+v_2}{\sigma-v_2},$$

then

$$
\begin{aligned}
v_1 \oplus v_2 &= \sigma \frac{\dfrac{\sigma+v_1}{\sigma-v_1}\dfrac{\sigma+v_2}{\sigma-v_2}-1}{\dfrac{\sigma+v_1}{\sigma-v_1}\dfrac{\sigma+v_2}{\sigma-v_2}+1} \\
&= \sigma \frac{[\sigma+v_1][\sigma+v_1]-[\sigma-v_1][\sigma-v_2]}{[\sigma+v_1][\sigma+v_1]+[\sigma-v_1][\sigma-v_2]} \\
&= \sigma^2 \frac{v_1+v_2}{\sigma^2+v_1 v_2},
\end{aligned}
$$

thus

$$
v_1 \oplus v_2 = \frac{v_1+v_2}{1+v_1 v_2/\sigma^2},
$$

as the velocity composition equation. As $\zeta(v) = \ln\left(\sqrt{\dfrac{\sigma+v}{\sigma-v}}\right)$, we obtain

$$
\exp(\phi\zeta/\sigma) = \sqrt{\frac{\sigma+v}{\sigma-v}}^{\phi/\sigma}. \tag{45}
$$

The general co-ordinate transformation with the two parameters ϕ and σ can be written as

$$
\left[
\begin{aligned}
t' &= \sigma \frac{\sqrt{\sigma+v}^{\phi/\sigma-1}}{\sqrt{\sigma-v}^{\phi/\sigma+1}}\left[t-\frac{v}{\sigma^2}x\right] \\
x' &= \sigma \frac{\sqrt{\sigma+v}^{\phi/\sigma-1}}{\sqrt{\sigma-v}^{\phi/\sigma+1}}\left[x-vt\right] \\
y' &= \frac{\sqrt{\sigma+v}^{\phi/\sigma}}{\sqrt{\sigma-v}^{\phi/\sigma}}y \\
z' &= \frac{\sqrt{\sigma+v}^{\phi/\sigma}}{\sqrt{\sigma-v}^{\phi/\sigma}}z
\end{aligned}
\right. \tag{46}
$$

The Lorentz transformations are obtained once $\phi \to 0$. But what is the effect of a non-zero ϕ on physical laws?